ON THE THEORETICAL FRACTURE STATISTICS OF THE HERTZ INDENTATION TEST

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ABSTRACT

The cumulative and local fracture probabilities of the Hertz indentation test were determined. When the material is subjected to the Hertz indentation test with a sphere indenter the stress field used for calculations depends on where the cracks originate the fracture, i.e. if the cracks begin in a zone within the contact between the material surface and sphere indenter or if cracks begin outside of such contact zone. Using the defined functions method for specific risk of Weibull's two or three-parameter fracture functions, the Evans functions and the cumulative and local fracture probabilities were determined for each zone. Additionaly, using the integral equation method the respective specific risk of Weibull's fracture functions were determined for cumulative and local fracture probabilities too. For the cumulative fracture probability the respective integral equation was solved by simple differentiation and its solution was a finite – integral – differential operator applied to the Evans function. For the local fracture probability two integral equations considering the fractures initiated within or outside the contact zone were solved. The last two integral equations were solved using a finite – differential operator applied to each function which can be obtained from the solution of an ordinary differential equation of second order with variable coefficients.

INTRODUCTION

The classical problem of the contact between a spherical surface and a flat plate was solved by Hertz [1] almost at the end of the nineteenth century, assuming elastic behaviour for the materials, the indenter and the plate. Many years later Frank and Lawn [2] proposed a theoretical analysis for the Hertzian fracture initiation based on Griffith´s criterion [3], using the energy balance, and the Hertz-Huber's stress field [1,4]. More about fundamental aspects of Hertzian contact can be found in [5,6].

The study of the Hertzian fracture through indentation of a hard sphere loaded onto a flat specimen it is interesting considering both theoretical and practical points of view.

From a theoretical or scientifical standpoint it is possible to develop models to understand the fracture behaviour for brittle materials. On the other hand, from the practical or technological viewpoint Hertzian fracture is of interest because it is related, for example, to friction and wear, ball milling and ballistic impact, among others. One of the principal advantages of the Hertz indentation test is that it allows us to determine the fracture toughness of brittle materials by means of this simple test [7–9]. Using the methodology proposed in reference [2] an extended approach with crack initiation under Hertzian sliding contac can be found in [10].

The statistical fracture mechanics, also known as probabilistic strength of materials was proposed by Weibull in 1939 [11] in order to take into account the scatter observed experimentaly in the fracture strength of brittle materials. At first his formulation was identified as a weakest-link-theory and associated with a simil of a chain where the fracture originates in the weakest link. At present, it is not necessary to use such simil and moreover, Weibull's theory has been used in a wide range of materials from brittle materials to ductile materials, considering brittle-ductile transition too. After Weibull's work many papers have been published using his formalism. The main objective is to determine the cumulative fracture probability for some material subjected to a constant or variable stress field, employing a specific risk of Weibull's function with a certain number of parameters which can be determined experimentally for a set of samples manufactured in the same manner and subjected to equal stress field. Kittl and Díaz [12] have published a paper in which they discussed differents materials and determined the Weibull parameters of the specific risk of Weibull´s function using statistical estimation methods and determining its respectives dispersions through the Fisher information method. A graphical method to obtain the Weibull parameters has also been developed in [12] and another approach considering diverse geometrical samples subjected to diverse stress field was published by Matthews et.al. [13] and by Evans and Jones [14]. In their work they introduced a method to determine the specific risk of Weibull's function, which Kittl later named Integral Equation Method, and solved the integral equations directly without the introduction of intermedia functions [15]. A more general treatment about the integral equation method applied to Weibull's fracture statistics was developed by Kittl and Díaz [16]. Special applications of such method in torsion and flexure were treated too [17-20]. In the case of Hertzian fracture Weibull's approach has been used widely [4,13,21-26].

The local fracture probability can be determined along with the determination of the cumulative fracture probability using Weibull's distribution function. In this sense the main idea is to determine the volumetric percentage of fractures initiated at a given point in some material subjected to certain stress field. Usually the defined method function is used to determine the parameters of the specific risk Weibull function. Most authors have used the two-parameter specific risk Weibull function. There are only two differents approaches dealing with this matter so far. The first approach was proposed by Oh and Finnie [21] and the second by Kittl and Camilo [27]. Both approaches are coincident when using a Weibullian two-parameter specific risk function but differ when a three-parameter Weibull function is used. Afterwards Trustrum [28] using the Oh and Finnie approach determined a Weibull modulus, one of the Weibull parameters, from bending, employing both fracture position and failure stress. The controversial approaches were discussed by Díaz et.al. [29]

showing by means of a numerical evaluation the differences in the results, when some material is subjected to flexure, using a specific risk of Weibull's three-parameter function, where Kittl and Camilo's formulation present a better fit than Oh and Finnie's formulation.

For the Hertz indentation test Oh and Finnie [21] applied their formalism to determine the local fracture probability obtaining the distribution of cone crack location for a three – parameter Weibull distribution for failure stress. Later Trustrum [26], using the same method developed a less complicated theoretical analysis to obtain some predictions for the distributions of the critical load and cone crack radius for a Weibullian two-parameter specific risk function. The application of Weibull statistics is due to a scatter of values observed for the cone crack radius, for the contact radius and for the critical load.

The objective of the present work is to determine both the cumulative fracture probability and the local fracture probability for some material subjected to Hertz indentation test using the specific risk of Weibull´s two and three – parameter functions and apply the integral equation method to find a more general specific risk of Weibull´s functions.

THE WEIBULL DISTRIBUTION FUNCTION

The cumulative probability distribution function of fracture strength for a brittle material subjected to a variable uniaxial stress field considering volume brittleness is given by [11,12]:

$$F(\sigma) = 1 - \exp\left\{-\frac{1}{V_0}\int_V \phi\left[\sigma(r)\right] dV\right\} \tag{1}$$

where V_o is the unity volume, r is the vector position, V is the volume of material subjected to stress, $\sigma(r) \leq \sigma$ is the variable stress field, σ is the maximum strength, F is the cumulative fracture probability and ϕ is the specific risk of Weibull's function. The variable stress field can be expressed as follows:

$$\begin{aligned} \sigma(r) &= \sigma f(r) \\ |f(r)| &\leq 1 \end{aligned} \tag{2}$$

where f is a function obtained from the elasticity theory which depends only on vector position and is minor or equal to one.

Considering equation (2) and rewritten equation (1) after Evans and Jones [14] the following equation is obtained:

$$\xi(\sigma) = \ln\frac{1}{1 - F(\sigma)} = \frac{1}{V_0}\int_V \phi\left[\sigma f(r)\right] dV \tag{3}$$

where $\xi(\sigma)$ is called Evan's function. This simplified notation allows for an easier mathematical treatments.

Weibull proposed two analytical expressions for the specific risk of function taking into account numerous experimental results over different materials subjected to different stress fields, and the best fit was obtained with the two or three-parameter functions. When the specific risk function has two parameters its expression is given by:

$$\phi(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^m \tag{4}$$

where m and σ_o are the Weibull parameters, m is known as Weibull's modulus and σ_o is a normalization parameter. On the other hand when the specific risk function has three parameters its respective expression adopted the following form:

$$\phi(\sigma) = \begin{cases} \left(\frac{\sigma - \sigma_L}{\sigma_0}\right)^m & \sigma \geq \sigma_L \\ 0 & \sigma < \sigma_L \end{cases} \tag{5}$$

where m and σ_o have the same sense than in equation (4) and σ_L the third parameter is the stress under which the fracture does not occur or, equivalently, when the cumulative fracture probability is equal to zero. Weibull's parameters depend on the manufacturing process of the material. Both previous equations can be used directly in equation (1) or (3) to obtain the cumulative fracture probability and its respective Weibull parameters. This method is called the defined equations method.

There is a second method to apply in the probabilistic strength of materials where the aim is precisely to obtain a general expression for the specific risk of Weibull functions where it is not necessary to postulate the existence of a certain number of parameters. That method is known as integral equation method. Equation (3) can be considered as an integral equation where function ϕ is an unknown function which must be determined because the stress field is known through the elasticity theory and $\xi(\sigma)$ is known by means of experimental data. For each specific stress field it is necessary to solve the integral equation (3). In some cases the solution can be determined by applying a finite operator over Evan's function i.e. by means of a finite number of mathematical operations and the solution can be expressed as an integro-differential form. In other cases the solution can not be solved as in the previous case and a infinite operator must be applied over Evan's function. When it is possible to apply Taylor's series expansions the specific risk of Weibull's function can be expressed as an infinite differential operator. Expanding $\xi(\sigma)$ and $\phi(\sigma)$ in Taylor's series we obtain:

$$\xi(\sigma) = \sum_{n=0}^{\infty} \xi^{(n)}(0)\frac{\sigma^n}{n!}$$
$$\phi(\sigma) = \sum_{n=0}^{\infty} \phi^{(n)}(0)\frac{\sigma^n}{n!} \tag{6}$$

Introducing equations (6) in equation (3) we obtain an equation with two series and both are equals when its respective coefficients are equal, so:

$$\xi^{(n)}(0) = \frac{\phi^{(n)}(0)}{V_0}\int_V \left[f(r)\right]^n dV \tag{7}$$

Equation (7) allows us to determine $\phi^{(n)}$ (0) which replaced into expression of $\phi(\sigma)$ in equation (6) gives:

$$\phi(\sigma) = V_0 \sum_{n=0}^{\infty} \frac{\xi^{(n)}(0)}{n!\int_V \left[f(r)\right]^n dV}\,\sigma^n \tag{8}$$

Equation (8) is a general solution of equation (3) when it is employed as an integral equation. Note that the solution is an infinite differential operator applied over Evan's function which is known through experimental data.

THE HERTZ INDENTATION STRESS FIELD

If a hard spherical indenter loaded onto a plane surface of a brittle specimen with surface brittleness the circunferential stress is compressive over the entire contact surface, only the radial stress must be considered to determine the fracture probability. According to the Hertz theory [1,4] the stress field distribution on the surface is non-uniform and two zones must be considered, within the contact area and outside the contact area. Within the contact area the surface of the material loaded is under compression. However, during the submergence under the indenter within the contact circle the maximum radial tensile stress experienced by the plane surface of the material subjected to load P is:

$$\sigma(r) = \frac{br}{cR}$$
$$0 < r < a \quad ; \quad 0 \leq \theta \leq 2\pi \tag{9}$$

where R is the radius of the sphere indenter, a is the radius of the contact circle, r is the radial distance from the centre of the contact circle, which define the radius of the cone crack when the crack ring is initiated, θ is an angle in polar coordinates. Constants b and c have the following expression:

$$
\begin{aligned}
b &= \frac{1}{2\pi}(1 - 2\nu) \\
c &= \frac{3}{4}\left[\frac{1 - \nu^2}{E} + \frac{1 - \nu'^2}{E'}\right]
\end{aligned}
\tag{10}
$$

In equation (10) v', E' and v, E are Poisson's ratio and Young's modulus of the sphere and specimen loaded, respectively. Outside the contact area the surface radial stress field is tensile and is given by:

$$
\begin{aligned}
&\sigma(r) = \frac{bP}{r^2} \\
&a \le r < \infty \;\; ; \;\; 0 \le \theta \le 2\pi
\end{aligned}
\tag{11}
$$

Maximum tensile stress σ in the specimen occurs on the surface at r = a, the border of the contact area, using equation (9) this maximum is:

$$
\sigma(r = a) = \frac{ba}{cR} = \sigma \tag{12}
$$

From equation (11) it is possible to obtain the maximun tensite stress, and is given by:

$$
\sigma(r = a) = \frac{bP}{a^2} = \sigma \tag{13}
$$

As both results in equation (12) and (13) must be equal, then the expression of the contact circle radius can be determined :

$$
a = (cRP)^{1/3} \tag{14}
$$

Equation (14) shows when an elastic spherical indenter of radius R is pressed with load P onto the surface of a semi-infinite elastic solid, a circular contact area increases in size with an increasing load and, in addition, depends on Poisson' ratio and Young's modulus for the indenter and the semi-infinite solid through constant c given by equation (10).

Thus, the tensile stress field on the surface of the specimen can be expressed in terms of the maximum tensile stress given by equation (13). Then, taking into account equation (9) and (11) the tensile stress field is:

$$
\sigma(r) = \begin{cases} \dfrac{r}{a}\sigma \le \sigma = \dfrac{bP}{a^2} & 0 < r < a \;\; ; \;\; 0 \le \theta \le 2\pi \\ \left(\dfrac{a}{r}\right)^2 \sigma \le \sigma = \dfrac{bP}{a^2} & a \le r < \infty \;\; ; \;\; 0 \le \theta \le 2\pi \end{cases} \tag{15}
$$

for within and outside contact area of the specimen, respectively.

THE CUMULATIVE PROBABILITY OF HERTZ INDENTATION FRACTURE

In the case of the Hertz indentation test the cracks are initiated on the surface of the specimen subjected to indenter, then the brittleness of the surface must be considered and in the cumulative probability of fracture the volume must be changed by surface. After that the Evans function given by equation (3) is writen as follows:

$$\xi(\sigma) = \ln\frac{1}{1 - F(\sigma)} = \frac{1}{S_0}\int_S \phi\left[\sigma f(r)\right] dS \qquad (16)$$

where S_o is the surface unity, S is the surface subjected to stress field and ϕ is the specific risk of Weibull surface brittleness function.

Defined function method for cumulative probability

When the specific Weibull risk function has two parameters as shown in equation (4) and taking into account the stress field given by equation (15), then the Evans function adopts the following expression:

$$\begin{aligned}\xi(\sigma) &= \frac{1}{S_0}\int_0^{2\pi}\int_0^{a}\left(\frac{r}{a}\frac{\sigma}{\sigma_0}\right)^m r\,dr\,d\theta + \frac{1}{S_0}\int_0^{2\pi}\int_a^{\infty}\left[\left(\frac{a}{r}\right)^2\frac{\sigma}{\sigma_0}\right]^m r\,dr\,d\theta \\ &= \frac{3\pi a^2}{S_0}\frac{m}{(m + 2)(m - 1)}\left(\frac{\sigma}{\sigma_0}\right)^m \qquad ; \qquad m \neq 1\end{aligned} \qquad (17)$$

Equation (17) allows for the determination of Weibull parameters through the Weibull diagram which is made by drawing ln $\xi(\sigma)$ versus ln σ. Taking logaritms to both sides of equation (17) and re-ordering:

$$\ln\xi(\sigma) = m\ln\sigma + \ln\left[\frac{3\pi a^2}{S_0}\frac{m}{(m + 2)(m - 1)\sigma_0^m}\right] \qquad (18)$$

With a series of experimental values to σ we can find the best fit by means, for example, of the minimun square method, obtaining Weibull modulus m as a slope of the straight line, and σ_o can be determined through the free term because the terms in square bracket in equation (18) are known, except σ_o. Note that $\xi(\sigma)$ is known through experimental values.

When the specific Weibull risk function has three-parameters as shown in equation (5), in order to determine the respective Evans's function it is neccessary to define a geometrical region where the integration must be made. The restrictions on the stress field indicated in equation (5) induce a geometrical restriction over the specimen subjected to the stress field. From equations (5) and (15) there are two regions where the stress field is valid. Within the contact area between the indenter sphere and specimen loaded the stress field restriction is:

$$\frac{r}{a}\sigma \ \geq \ \sigma_L \tag{19}$$

which implies the following geometrical restriction:

$$\frac{\sigma_L}{\sigma} a \ \leq \ r \ \leq \ a \tag{20}$$

Consequently, outside the contact area the respective stress restriction is:

$$\left(\frac{a}{r}\right)^2 \sigma \ \geq \ \sigma_L \tag{21}$$

which implies the following geometrical restriction in this case:

$$a \ \leq \ r \ \leq \ a\sqrt{\frac{\sigma}{\sigma_L}} \tag{22}$$

Equations (20) and (22) show that there are two regions where the cracks initiated promote the Hertzian fracture, one within the contact area and another one outside the contact area of the specimen subjected to the Hertz indentation test. Not all surface contributes to Hertzian fracture. Then, with equations (3), (5), (15), (20) and (22) the Evans function is given by:

$$\xi(\sigma) = \frac{2\pi}{S_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m \left[\int_{\frac{\sigma_L}{\sigma}a}^{a}\left(\frac{r}{a}\frac{\sigma}{\sigma_L} - 1\right)^m r\,dr + \int_{a}^{a\sqrt{\frac{\sigma}{\sigma_L}}}\left[\left(\frac{a}{r}\right)^2\frac{\sigma}{\sigma_L} - 1\right]^m r\,dr\right] \tag{23}$$

Introducing the following changes of variables:

$$\eta = \frac{r}{a}\frac{\sigma}{\sigma_L} \quad ; \quad \xi = \left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L} \tag{24}$$

Equation (23) is finally transformed into:

$$\xi(\sigma) = \frac{2\pi a^2}{S_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m \left[\begin{array}{l} \dfrac{1}{(m+1)(m+2)}\left(\dfrac{\sigma_L}{\sigma}\right)^2\left(\dfrac{\sigma}{\sigma_L} - 1\right)^m\left[1 + (m+1)\dfrac{\sigma}{\sigma_L}\right] \\ + \dfrac{1}{2}\dfrac{\sigma}{\sigma_L}\displaystyle\int_1^{\sigma/\sigma_L}\dfrac{(\eta - 1)^m}{\eta^2}d\eta \end{array}\right] \tag{25}$$

Using equation (25) the Weibull parameters can be obtained and differents methods have been developed for them, for example linear regression, maximum likelihood, minimun chi-square. Kittl et.al. [30] have discussed such methods including a numerical simulations approach. However there is an easier method called nomographic method which, preparing a non-dimensional nomogram allows us to determine the Weibull parameters. Such nomographical method has been developed by Kittl and León [31]. In general, the equation (25) can be expressed as follows:

$$\xi(\sigma) = \frac{S}{S_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m \psi\left(\frac{\sigma}{\sigma_L}, m\right) \tag{26}$$

where $\psi(\sigma/\sigma_L, m)$ is a function given by:

$$\psi\left(\frac{\sigma}{\sigma_L}, m\right) = \frac{1}{(m+1)(m+2)}\left(\frac{\sigma_L}{\sigma}\right)^2\left(\frac{\sigma}{\sigma_L} - 1\right)^m\left[1 + (m+1)\frac{\sigma}{\sigma_L}\right] + \frac{1}{2}\frac{\sigma}{\sigma_L}\int_1^{\sigma/\sigma_L}\frac{(\eta-1)^m}{\eta^2}d\eta \tag{27}$$

Taking logarithm to the multiplicative factor of function $\psi(\sigma/\sigma_L, m)$ and denominated C which has the following expression:

$$C = \ln\left[\frac{S}{S_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m\right] \tag{28}$$

Considering equations (27) and (28) and taking logarithm in both side of equation (26) it is transformed into:

$$\ln\xi(\sigma) = \ln\psi\left(\frac{\sigma}{\sigma_L}, m\right) + C \tag{29}$$

Now, if we plot $\ln\psi(\sigma/\sigma_L, m)$ against ln σ/σ_L for several values of σ/σ_L and m we obtain a non-dimensional diagram resulting differents curves. From the experimental point of view the Evans functions $\xi(\sigma)$ can be known for a set of experimental values. In that case the Weibull diagram ln ξ (σ) against lnσ must be made. The distributions of the experimental points is fitted to a curve of the nomogram by means of a superposition of the experimental diagram, ln $\xi(\sigma)$ against lnσ, and this allows for parameter m to be determined. Note that this fit is equivalent to linear traslation where the nomogram is moved on Weibull diagram. Equation (29) corresponds to that traslation. Once time parameter m is known the others parameters can be determined by reading the coordinates of the origin of the nomogram in respect to the Weibull diagram. Then, the distance between the axes ln $\xi(\sigma)$ and $\ln\psi(\sigma/\sigma_L, m)$ allows for the determination of σ_L and finally the distance C between the axes $\ln\sigma$ and $\ln\sigma/\sigma_L$ allows for the determination of σ_o in accordance with equation (28) due to C being known directly from the diagrams, m and σ_L were previously determined, S_O is the unity of surface and S is the surface subjected to stress field, both known too. This estimation method was used by Díaz and Morales [18] to determine the Weibull parameters in glass cylinders subjected to torsion.

Integral equation method for cumulative probability

It was said that equation (3) can be considered as an integral equation where the specific risk of Weibull's function is an unknown function to determine. Then, according to the Evans function given by equation (3) and considering the stress field defined in equation (15) for tensile stress within and outside the contact surface in a Hertz indentation test, the following equation can be writen:

$$\xi(\sigma) = \frac{1}{S_0}\int_0^{2\pi}\int_0^{a}\phi\left(\frac{r}{a}\sigma\right)rdrd\theta + \frac{1}{S_0}\int_0^{2\pi}\int_a^{\infty}\phi\left[\left(\frac{a}{r}\right)^2\sigma\right]rdrd\theta \qquad (30)$$

where the first term at the right side of equation (30) takes into account the crack initiated whithin the contact area between the spherical indenter and the specimen, and the second term considered the crack initiated outside the contact area. Given the following changes of variables in equation (29):

$$\eta = \frac{r}{a}\sigma \qquad ; \qquad \xi = \left(\frac{a}{r}\right)^2\sigma \qquad (31)$$

After some manipulation and rearranging equation (30) is transformed into:

$$\frac{S_0}{2\pi a^2}\sigma^2\xi(\sigma) = \int_0^{\sigma}\phi(\eta)\eta d\eta + \frac{1}{2}\sigma^3\int_0^{\sigma}\frac{\phi(\eta)}{\eta^2}d\eta \qquad (32)$$

From this equation (32) it is easy to see that the method to solve it is by simple differentiation. Then, differentiating once and rearranging equation (32) gives:

$$\frac{S_0}{3\pi a^2}\frac{1}{\sigma^2}\frac{d}{d\sigma}\left[\sigma^2\xi(\sigma)\right] = \frac{1}{\sigma}\phi(\sigma) + \int_0^{\sigma}\frac{\phi(\eta)}{\eta^2}d\eta \qquad (33)$$

Differentiating once again and rewriting we obtain the following differential equation:

$$\phi'(\sigma) = \frac{d\phi(\sigma)}{d\sigma} = \frac{S_0}{3\pi a^2}\sigma\frac{d}{d\sigma}\left\{\frac{1}{\sigma^2}\frac{d}{d\sigma}\left[\sigma^2\xi(\sigma)\right]\right\} \qquad (34)$$

Now, integrating equation (34) it is possible to determine the specific risk of Weibull's function, that is to say:

$$\phi(\sigma) = \int_0^{\sigma} \frac{d\phi(\sigma)}{d\sigma} d\sigma \tag{35}$$

Finally, the solution to the integral equation given by equation (30) is the following:

$$\phi(\sigma) = \frac{S_0}{3\pi a^2} \int_0^{\sigma} \frac{d}{d\eta} \left\{ \frac{1}{\eta^2} \frac{d}{d\eta} \left[\eta^2 \xi(\eta) \right] \right\} \eta d\eta \tag{36}$$

Note that equation (36) is a finite integral-differential operator applied to Evans's function $\xi(\sigma)$ which is known through experimental values. Equation (36) is an exact solution to obtain the specific risk of Weibull’s function for the case of Hertzian indentation test for a specimen with surface brittleness subjected to Hertz contact. An approximate solution can be obtained by using the method of Taylor’s series expansion like equation (8). In such situation if we considered equation (6) and replace Taylor’s serie expansion to Evan’s function, $\xi(\sigma)$, in equation (36) the result is given by:

$$\phi(\sigma) = \frac{S_0}{3\pi a^2} \sum_{n=2}^{\infty} \frac{(n+2)(n-1)}{n} \xi^{(n)}(0) \frac{\sigma^n}{n!} \tag{37}$$

which requires the experimental values of Evans’s function too and its corresponding evaluation of derivates of $\xi(\sigma)$ evaluated in σ equal to zero. In equation (37) the values of n must be non equal to zero and non equal to one due to its non convergence. The same solution given by equation (37) can be obtained from integral equation (30) following the method explained in equations (6), (7) and (8). Obviously, if we replace the Evans function from equation (17) in equation (36) we can obtain the two – parameter Weibull´s function as a particular case.

If the Hertz indentation test is applied until the appearance of the first fracture then stress σ and point r in which it appears can be determined, as well as the cumulative probability of Hertzian fracture. The initiation of the first crack can be detected by sound, and then through observation with a microscope to determine if the crack was initiated within or outside the contact surface. However, if applied load P is fixed and produces more than one crack, only local fracture probability can be used, determining the point r where the crack is initiated and separating both cases when the crack is initiated whithin the contact area or outside for a brittle specimen subjected to Hertz indentation test.

THE LOCAL PROBABILITY OF HERTZ INDENTATION FRACTURE

The two approaches respect to what formalism must be used to determine the local probability of fracture were discussed in the introduction of this paper. In accordance with that we use the Kittl and Camilo approach [27] here. The same formalism was used by Díaz et.al. [32] to determine both cumulative and local fracture probabilities of glass cylinders subjected to flexure.

The local fracture probability is given by:

$$\frac{n(V')}{n(V)} = \frac{\int_{V'} \phi\left[\sigma(r)\right]dV}{\int_{V} \phi\left[\sigma(r)\right]dV} = \frac{\xi(\sigma,V')}{\xi(\sigma,V)} \tag{38}$$

where V' is a volume included in V where the cracks are initiated, n(V') is the number of fractures which are initiated in volume V' and n(V) is the total number of fractures. Equation (38) yields the fraction or percentage of fractures in the volume V' of the specimen at the stress σ. Due to the Hertz indentation test the fractures occur at the surface of the material loaded, then we must use surface S instead volume V. After that, the local probability of Hertzian indentation test is given by:

$$\frac{n(S')}{n(S)} = \frac{\int_{S'} \phi\left[\sigma(r)\right]dS}{\int_{S} \phi\left[\sigma(r)\right]dS} = \frac{\xi(\sigma,S')}{\xi(\sigma,S)} \tag{39}$$

Here it is necessary to treat both the fractures initiated within the contact surface and the fractures initiated outside the contact surface separately.

Defined function method for local probability

Employing a specific risk of Weibull's function the local probability of Hertzian fracture with a two-parameter function initiated within the contact surface, zone 1, between spherical indenter and specimen loaded and considering the stress field in accordance with equation (15), $0 \le r \le a$, is given by:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_1(r,\sigma)}{\xi_1(\sigma)} = \frac{\int_0^{2\pi}\int_0^{r}\left(\frac{r}{a}\frac{\sigma}{\sigma_0}\right)^m rdrd\theta}{\int_0^{2\pi}\int_0^{a}\left(\frac{r}{a}\frac{\sigma}{\sigma_0}\right)^m rdrd\theta} \tag{40}$$

and evaluating the integrals in equation (40) we obtain:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_1(r,\sigma)}{\xi_1(\sigma)} = \left(\frac{r}{a}\right)^{m+2} = \frac{n(r)}{n} \tag{41}$$

Note that equation (41) does not depend on both Weibull parameter σ_0 and stress σ.

In equation (41) the left side is known through experimental values and we can make a Weibull diagram. Taking logarithm in both side of equation (41) follows:

$$\ln\frac{n(r)}{n} = (m + 2)\ln\frac{r}{a} \tag{42}$$

Now, plotting ln(n(r)/n) against ln(r/a) we can estimate Weibull parameter m because the slope of the Weibull diagram for a set of experimental values is equal to (m+2).

When the fractures are initiated outside the contact surface, zone2, and considering the stress field in accordance with equation (15), $a \le r < \infty$, the local probability of Hertzian fracture is given by:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_2(r,\sigma)}{\xi_2(\sigma)} = \frac{\int_0^{2\pi}\int_a^{r}\left[\left(\frac{a}{r}\right)^2\frac{\sigma}{\sigma_0}\right]^m r\,dr\,d\theta}{\int_0^{2\pi}\int_a^{\infty}\left[\left(\frac{a}{r}\right)^2\frac{\sigma}{\sigma_0}\right]^m r\,dr\,d\theta} \tag{43}$$

evaluating the integrals in equation (43) it is transformed into:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_2(r,\sigma)}{\xi_2(\sigma)} = 1 + \left(\frac{a}{r}\right)^{2m-2} = \frac{n(r)}{n} \tag{44}$$

Like equation (41) this equation (44) is independent from Weibull parameter σ_0 and stress σ. Rearranging equation (44) and taking logarithm in both side of equation follows:

$$\ln\left[1 - \frac{n(r)}{n}\right] = 2(m - 1)\ln\frac{a}{r} \tag{45}$$

Then, plotting ln[1–n(r)/n] against ln(a/r) we can estimate the Weibull parameter m through determining the slope of the Weibull diagram for a set of experimental values, which is equal to (2m – 1) .

When we use a specific risk of Weibull's three-parameter function it is necessary to define a geometrical region where the fractures can be initiated. For the first zone, that is to say within the contact surface for the Hertz indentation test, the first zone continues to be defined by equation (20). Then, considering equation (15) and (20) the local probability of Hertzian fractures is given by:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_1(r,\sigma)}{\xi_1(\sigma)} = \frac{\int_0^{2\pi}\int_{a\frac{\sigma_L}{\sigma}}^{r}\left(\frac{\sigma_L}{\sigma_0}\right)^m\left[\frac{r}{a}\frac{\sigma}{\sigma_L} - 1\right]^m rdrd\theta}{\int_0^{2\pi}\int_{a\frac{\sigma_L}{\sigma}}^{a}\left(\frac{\sigma_L}{\sigma_0}\right)^m\left[\frac{r}{a}\frac{\sigma}{\sigma_L} - 1\right]^m rdrd\theta} \quad (46)$$

Introducing in equation (46) the variable change $\eta = (r/a)(\sigma/\sigma_L)$ given by equation (24), then equation (46) is transformed into:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_1(r,\sigma)}{\xi_1(\sigma)} = \frac{\int_1^{\frac{r}{a}\frac{\sigma}{\sigma_L}}(\eta - 1)^m \eta d\eta}{\int_1^{\frac{\sigma}{\sigma_L}}(\eta - 1)^m \eta d\eta}$$
$$= \frac{\frac{r}{a}\left(\frac{\sigma}{\sigma_L}\frac{r}{a} - 1\right)^{m+1}\left[1 + \frac{1}{m+1}\frac{a}{r}\frac{\sigma_L}{\sigma}\right]}{\left(\frac{\sigma}{\sigma_L} - 1\right)^{m+1}\left[1 + \frac{1}{m+1}\frac{\sigma_L}{\sigma}\right]} = \psi\left(\frac{\sigma}{\sigma_L},\frac{r}{a},m\right) \quad (47)$$

where $\psi\left(\frac{\sigma}{\sigma_L},\frac{r}{a},m\right)$ is a non–dimensional function. Note that equation (47) is independent from Weibull parameter σ_0. In order to determine Weibull parameters m and σ_L we can plot (n (r,σ)/n) against (r/a) where the left side of equation (47) is known through experimental values. However it is necessary to use a more complex method than the Weibull two-parameter function employed from equation (41). It is posible to make a numerical method similar to that developed from equation (26), in order to obtain a non–dimensional diagram with different curves and select that better fittness with a curve determined for a set of experimental data which can determine the Weibull parameters.

For the second zone, zone 2, outside contact surface of Hertz indentation test and using a specific risk of Weibull three-parameter function, as in the previous case there is a geometrical region where the fracture occurs and it is defined by equation (22). Then, considering equation (15) and equation (22) the local probability of Hertzian fracture is given by:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_2(r,\sigma)}{\xi_2(\sigma)} = \frac{\int_0^{2\pi}\int_r^{a\sqrt{\frac{\sigma}{\sigma_L}}} \left(\frac{\sigma}{\sigma_0}\right)^m \left[\left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L} - 1\right]^m r dr d\theta}{\int_0^{2\pi}\int_a^{a\sqrt{\frac{\sigma}{\sigma_L}}} \left(\frac{\sigma}{\sigma_0}\right)^m \left[\left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L} - 1\right]^m r dr d\theta} \tag{48}$$

Introducing in equation (48) the variable change $\xi = (a/r)^2\ (\sigma/\sigma_L)$ given by equation (24), then equation (48) is transformed into:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_2(r,\sigma)}{\xi_2(\sigma)} = \frac{\int_1^{\left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L}} \frac{(\eta - 1)^m}{\eta^2} d\eta}{\int_1^{\sigma/\sigma_L} \frac{(\eta - 1)^m}{\eta^2} d\eta} \tag{49}$$

Note that equation (49) is independent from Weibull parameter σ_o. Rewriting equation (49) it is transformed as follows:

$$\frac{n(r,\sigma)}{n} = \frac{\xi_2(r,\sigma)}{\xi_2(\sigma)} = \frac{\int_1^{\left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L}} \frac{(\eta - 1)^{m-1}}{\eta} d\eta - \frac{1}{m}\left(\frac{r}{a}\right)^2 \frac{\sigma_L}{\sigma}\left[\left(\frac{a}{r}\right)^2 \frac{\sigma}{\sigma_L} - 1\right]^m}{\int_1^{\sigma/\sigma_L} \frac{(\eta - 1)^{m-1}}{\eta} d\eta - \frac{1}{m}\frac{\sigma_L}{\sigma}\left[\frac{\sigma}{\sigma_L} - 1\right]^m} \tag{50}$$

$$= \psi\left(\frac{\sigma}{\sigma_L}, \frac{r}{a}, m\right)$$

where $\psi\left(\frac{\sigma}{\sigma_L}, \frac{r}{a}, m\right)$ is a non–dimensional function. The same comments made about equation (47) are valid to equation (50). In order to determine Weibull's parameters m and σ_L we can plot (n $(r,\sigma)/n$) against (r/a) where the left side of equation (50) is known through experimental values. However the greater complexity of equation (50) than that obtained for a three-parameter Weibull function in equation (44) requires the use of numerical methods to determine the Weibull parameters. Consequently, it is posible to make a numerical method similar to that developed from equation (26), in order to obtain a non–dimensional diagram with different curves and select that better fittness with a curve determined for a set of experimental data.

Using the defined function method with a specific risk of two or three-parameter Weibull function we need to establish where the fractures are initiated in order to determine the local fracture probability for the Hertz indentation test. The load P applied with a spherical indenter can be varied for carrying out tests to obtain a set of experimental values for different load conditions. At the same time the size of the indenter can be changed in order to obtain both cumulative and local fracture probabilities.

Integral equation method for local probability

When in the local probability of fracture, given by equation (39) the specific risk of Weibull's function is unknown, then this equation must be treated as an integral equation. Now we solve the integral equation for a Hertz indentation test considering both cases, when the fractures are initiated within the contact surface and outside.

Equation (39) can be writen as follows:

$$\frac{n(r,\sigma)}{n}\int_{S}\phi\left[\sigma(r)\right]\,dS = \int_{S'}\phi\left[\sigma(r)\right]\,dS \qquad (51)$$

where $n(r,\sigma)/n$ is known through experimental values and ϕ is an unknown function which must be determined. Note that the fracture fraction has been expressed explicitly as function of stress σ given a more general treatment. Then, such fracture fraction depends on the position where the fracture initiated and the level of stress reached in that position.

When the fractures are initiated within the contact surface, previously referred to as zone 1, between spherical indenter and specimen loaded, and considering the stress field in accordance with equation (15) with $0 \le r < a$, then the integral equation of the local probability of Hertzian fracture, taking into account equation (51) is given by:

$$\frac{n(r,\sigma)}{n}\int_{0}^{2\pi}\int_{0}^{a}\phi\left(\frac{r}{a}\sigma\right)r\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{r}\phi\left(\frac{r}{a}\sigma\right)r\,dr\,d\theta \qquad (52)$$

First, integrating equation (52) respect to θ, then using the variable change $\eta = (r/a)\sigma$ given by equation (31) and differentiating partially respect to r and finally putting r = a, after some manipulation the integral equation (52) yields:

$$a\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\int_{0}^{\sigma}\phi(\eta)\eta\,d\eta = \sigma^2\phi(\sigma) \qquad (53)$$

In order to solve integral equation (53) we can differentiate partially respect to σ, hence this equation becomes:

$$a\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\int_{0}^{\sigma}\phi(\eta)\eta\,d\eta + a\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\sigma\phi(\sigma) = 2\sigma\phi(\sigma) + \sigma^2\phi'(\sigma) \qquad (54)$$

Rendering the new following integral equation:

$$\chi(\sigma) = \int_0^{\sigma} \phi(\eta)\eta d\eta \tag{55}$$

which obviously allows us to get $\phi(\sigma)$ by differentiation. Its solution is:

$$\phi(\sigma) = \frac{1}{\sigma}\frac{d}{d\sigma}\left[\chi(\sigma)\right] \tag{56}$$

Now, we must obtain the expression of function $\chi(\sigma)$ and the initial problem of the solution of the integral equation of local probability of Hertzian indentation test, for zone 1, within the contact surface, can be solved. Putting function $\chi(\sigma)$ defined in equation (55) in equation (54) the following is obtained

$$\sigma^2 \frac{d}{d\sigma}\left\{\frac{1}{\sigma}\frac{d}{d\sigma}\left[\chi(\sigma)\right]\right\} = a\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\chi(\sigma) + \left[a\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a} - 2\right]\frac{d}{d\sigma}\left[\chi(\sigma)\right] \tag{57}$$

Equation (57) can be rearranged as follows:

$$\chi''(\sigma) + \frac{1}{\sigma}\left[1 - a\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\right]\chi'(\sigma) - \frac{a}{\sigma}\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\chi(\sigma) = 0 \tag{58}$$

which is an ordinary differential equation of second order with variable coefficients and allows us to obtain function $\chi(\sigma)$. Then upon getting $\chi(\sigma)$ from equation (58), it is introduced in equation (56) to finally obtain the solution to integral equation (52). Thus, this is the solution for zone 1 where the fractures in the Hertzian indentation test are initiated within the contact area.

Introducing equation (41), which corresponds to the result of local fracture probability within the contact area when the specific risk of Weibull´s functions has two parameters, into the differential equation (58) yields:

$$\chi''(\sigma) - \frac{1}{\sigma}(m + 1)\chi'(\sigma) = 0 \tag{59}$$

The differential equation (59) can be solved easily by putting it as follows:

$$\frac{\chi''(\sigma)}{\chi'(\sigma)} = \frac{m + 1}{\sigma} \tag{60}$$

integrating this equation (60) and using an appropiate constant of integration we obtain:

$$\chi'(\sigma) = \frac{\sigma^{m+1}}{\sigma_0^m} \tag{61}$$

and integrating the solution once again in this particular case is given by:

$$\chi(\sigma) = \frac{1}{m+2}\sigma^2\left(\frac{\sigma}{\sigma_0}\right)^m \tag{62}$$

Then, replacing this equation (62), which is the solution to differential equation (59), in equation (56) it is obviously possible to obtain the specific risk of Weibull´s two-parameter function for local fracture probability when the specimen is subjected to the Hertz indentation test and the fractures are initiated within the contact area.

When the fractures are initiated in zone 2, that it to say outside the contact surface between spherical indenter and specimen loaded, and considering the stress field in accordance with equation (15) with $a \leq r < \infty$, then the integral equation of the local probability of Hertzian indentation test, taking into account equation (51) is given by:

$$\frac{n(r,\sigma)}{n}\int_0^{2\pi}\int_a^{\infty}\phi\left[\left(\frac{a}{r}\right)^2\sigma\right]rdrd\theta = \int_0^{2\pi}\int_a^{r}\phi\left[\left(\frac{a}{r}\right)^2\sigma\right]rdrd\theta \tag{63}$$

Integrating this equation (63) respect to θ and with variable change $\eta = (a/r)^2\,\sigma$ given by equation (31) , integral equation (63) is transformed into:

$$\frac{n(r,\sigma)}{n}\int_0^{\sigma}\phi(\eta)\frac{d\eta}{\eta^2} = \int_{\left(\frac{a}{r}\right)^2\sigma}^{\sigma}\phi(\eta)\frac{d\eta}{\eta^2} \tag{64}$$

Differentiating partially integral equation (64) respect to r in order to solve it and evaluating r = a this equation yields:

$$\frac{a}{2}\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\int_0^{\sigma}\phi(\eta)\frac{d\eta}{\eta^2} = \frac{1}{\sigma}\phi(\sigma) \tag{65}$$

If we differentiated partially integral equation (65) respect to σ and after some manipulation it becomes:

$$\frac{a}{2}\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\int_0^{\sigma}\phi(\eta)\frac{d\eta}{\eta^2} + \frac{1}{\sigma^2}\left[1 + \frac{a}{2}\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\right]\phi(\sigma) = \frac{1}{\sigma}\phi'(\sigma) \tag{66}$$

Let’s consider the new following integral equation:

$$\Psi(\sigma) = \int_0^\sigma \phi(\eta)\frac{d\eta}{\eta^2} \qquad (67)$$

which allows us to obtain $\phi(\sigma)$ by simple differentiation and its solution is:

$$\phi(\sigma) = \sigma^2 \frac{d}{d\sigma}\left[\Psi(\sigma)\right] \qquad (68)$$

Analogue comment made after equation (56) can be made about equation (68). This equation (68) is the solution to the integral equation (67) where we need to obtain the expression of function $\Psi(\sigma)$. Thus, the integral equation of local probability of Hertzian indentation test for zone 2, outside the contact surface, can be solved. Replacing the function $\Psi(\sigma)$ defined in equation (67) in integral equation (66) after some manipulations gives:

$$\begin{aligned}\frac{1}{\sigma}\frac{d}{d\sigma}\left\{\sigma^2 \frac{d}{d\sigma}\left[\Psi(\sigma)\right]\right\} &= \frac{a}{2}\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\Psi(\sigma) \\ &+ \left[1 + \frac{a}{2}\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\right]\frac{d}{d\sigma}\left[\Psi(\sigma)\right]\end{aligned} \qquad (69)$$

Finally, equation (69) can be rearranged and it becomes:

$$\Psi''(\sigma) + \frac{1}{\sigma}\left[1 - \frac{a}{2}\frac{\partial}{\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\right]\Psi'(\sigma) - \frac{a}{2}\frac{\partial^2}{\partial\sigma\partial r}\left\{\frac{n(r,\sigma)}{n}\right\}_{r=a}\Psi(\sigma) = 0 \qquad (70)$$

which is, like equation (58), an ordinary differential equation of second order with variable coefficients and allows us to obtain the function $\Psi(\sigma)$. Then introducing $\Psi(\sigma)$ obtained from equation (70) in equation (68) gives the solution of integral equation (64). After that it is possible to know the specific risk of Weibull's function for zone 2 where the fractures in the Hertz indentation test are initiated outside the contact area.

Introducing equation (44), which corresponds to the result of local fracture probability outside the contact area when the specific risk of Weibull´s functions has two parameters, into differential equation (70) yields:

$$\Psi''(\sigma) - \frac{1}{\sigma}(m - 2)\Psi'(\sigma) = 0 \qquad (71)$$

Differential equation (71) can be solved easily putting it as follows:

$$\frac{\Psi''(\sigma)}{\Psi'(\sigma)} = \frac{m - 2}{\sigma} \qquad (72)$$

integrating this equation (72) and using an appropiate constant of integration we obtain:

$$\Psi'(\sigma) = \frac{\sigma^{m-2}}{\sigma_0^m} \tag{73}$$

and integrating the solution once again in this particular case is given by:

$$\Psi(\sigma) = \frac{1}{m-1}\frac{1}{\sigma}\left(\frac{\sigma}{\sigma_0}\right)^m \tag{74}$$

Then, replacing this equation (74), which is the solution of differential equation (71), in equation (68) it is obviously possible to obtain the specific risk of Weibull´s two-parameter function for local fracture probability when the specimen is subjected to the Hertz indentation test and the fractures are initiated outside the contact area for m non equal to one.

Depending on where the cracks grow to initiate the fractures, within or outside the contact area, the general integral equations for local probabilities of fracture given by equations (52) and (64), for the Hertz indentation test, using a finite – differential operators applied to respective functions were solved in both cases. These functions were obtained from the solution of an ordinary differential equation of second order with variable coefficients obtained from equations (58) and (70). Being obtained, such functions were introduced in the finite – differential operators given by equations (56) and (68), respectively. The finite – differential operators are applied to the percentages of fractures initiated within or outside the contact area, respectively, n(r,σ)/n, which can be obtained from a set of experimental data. These functions follow the same role of the Evans function to solve the integral equation for cumulative fracture probability.

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